

# On Boundedness Problems for Pushdown Vector Addition Systems<sup>\*</sup>

Jérôme Leroux<sup>1</sup>, Grégoire Sutre<sup>1</sup>, and Patrick Totzke<sup>2</sup>

<sup>1</sup> Univ. Bordeaux & CNRS, LaBRI, UMR 5800, Talence, France

<sup>2</sup> Department of Computer Science, University of Warwick, UK

**Abstract.** We study pushdown vector addition systems, which are synchronized products of pushdown automata with vector addition systems. The question of the boundedness of the reachability set for this model can be refined into two decision problems that ask if infinitely many counter values or stack configurations are reachable, respectively. Counter boundedness seems to be the more intricate problem. We show decidability in exponential time for one-dimensional systems. The proof is via a small witness property derived from an analysis of derivation trees of grammar-controlled vector addition systems.

## 1 Introduction

Pushdown vector addition systems are finite automata that can independently manipulate a pushdown stack and several counters. They are defined as synchronized products of vector addition systems with pushdown automata. Vector addition systems, shortly *VAS*, are a classical model for concurrent systems and are computationally equivalent to Petri nets. Formally, a  $k$ -dimensional *vector addition system* is a finite set  $\mathbf{A} \subseteq \mathbb{Z}^k$  of vectors called *actions*. Each action  $\mathbf{a} \in \mathbf{A}$  induces a binary relation  $\xrightarrow{\mathbf{a}}$  over  $\mathbb{N}^k$ , defined by  $\mathbf{c} \xrightarrow{\mathbf{a}} \mathbf{d}$  if  $\mathbf{d} = \mathbf{c} + \mathbf{a}$ .

A  $k$ -dimensional *pushdown vector addition system*, shortly *PVAS*, is a tuple  $(Q, \Gamma, q_{\text{init}}, \mathbf{c}_{\text{init}}, w_{\text{init}}, \Delta)$  where  $Q$  is a finite set of *states*,  $\Gamma$  is a finite *stack alphabet*,  $q_{\text{init}} \in Q$  is an *initial state*,  $\mathbf{c}_{\text{init}} \in \mathbb{N}^k$  is an *initial assignment of the counters*,  $w_{\text{init}} \in \Gamma^*$  is an *initial stack content*, and  $\Delta \subseteq Q \times \mathbb{Z}^k \times \text{Op}(\Gamma) \times Q$  is a finite set of *transitions* where  $\text{Op}(\Gamma) \stackrel{\text{def}}{=} \{\text{push}(\gamma), \text{pop}(\gamma), \text{nop} \mid \gamma \in \Gamma\}$  is the set of stack operations. The *size* of VAS, PVAS (and GVAS introduced later) are defined as expected with numbers encoded in binary.

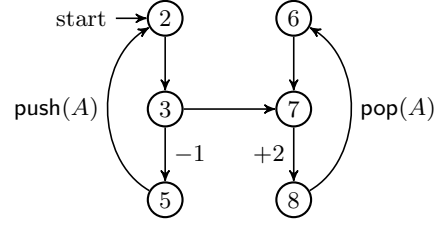
*Example 1.1.* Consider the program on the left of Figure 1, that doubles the value of the global variable  $x$ . The  $\star$  expression non-deterministically evaluates to a Boolean, as it is often the case in abstraction of programs [1]. On the right is a 1-dimensional PVAS that models this procedure: states correspond to lines in the program code, operations on the variable  $x$  are directly applied, and the call stack is reflected on the pushdown stack.  $\square$

<sup>\*</sup> This work was partially supported by ANR project REACHARD (ANR-11-BS02-001).

```

1:  $x \leftarrow n$ 
2: procedure DOUBLEX
3:   if ( $\star \wedge x > 0$ ) then
4:      $x \leftarrow (x - 1)$ 
5:     DOUBLEX
6:   end if
7:    $x \leftarrow (x + 2)$ 
8: end procedure

```



**Fig. 1.** A PVAS modeling a recursive program.

The semantics of PVAS is defined as follows. A *configuration* is a triple  $(q, \mathbf{c}, w) \in Q \times \mathbb{N}^k \times \Gamma^*$  consisting of a state, a vector of natural numbers, and a stack content. The binary *step* relation  $\rightarrow$  over configurations is defined by  $(p, \mathbf{c}, u) \rightarrow (q, \mathbf{d}, v)$  if there is a transition  $(p, op, \mathbf{a}, q) \in \Delta$  such that  $\mathbf{c} \xrightarrow{\mathbf{a}} \mathbf{d}$  and one of the following conditions holds: either  $op = \text{push}(\gamma)$  and  $v = u\gamma$ , or  $op = \text{pop}(\gamma)$  and  $u = v\gamma$ , or  $op = \text{nop}$  and  $u = v$ . The reflexive and transitive closure of  $\rightarrow$  is denoted by  $\rightarrow^*$ .

The *reachability set* of a PVAS is the set of configurations  $(q, \mathbf{c}, w)$  such that  $(q_{\text{init}}, \mathbf{c}_{\text{init}}, w_{\text{init}}) \xrightarrow{*} (q, \mathbf{c}, w)$ . The reachability problem asks if a given configuration  $(q, \mathbf{c}, w)$  is in the reachability set of a given PVAS. The decidability of this problem is open. Notice that for vector addition systems, even though the reachability problem is decidable [12,6], no primitive upper bound of complexity is known (see [9] for a first upper bound). However, a variant called the coverability problem is known to be EXPSpace-complete [13,11]. Adapted to PVAS, the coverability problem takes as input a PVAS and a state  $q \in Q$  and asks if there exists a reachable configuration of the form  $(q, \mathbf{c}, w)$  for some  $\mathbf{c}$  and  $w$ . The decidability of the coverability problem for PVAS is also open. In fact, coverability and reachability are inter-reducible (in logspace) for this class [7,10]. In dimension one, we recently proved that coverability is decidable [10].

Both coverability and reachability are clearly decidable for PVAS with finite reachability sets. These PVAS are said to be *bounded*. In [8], this class is proved to be recursive, i.e. the boundedness problem for PVAS is decidable. The complexity of this problem is known to be TOWER-hard [7]. The decidability is obtained by observing that if the reachability set of a PVAS is finite, its cardinality is at most hyper-Ackermannian in the size of the PVAS. Even though this bound is tight [8], the exact complexity of the boundedness problem is still open. Indeed, it is possible that there exist small certificates that witness infinite reachability sets. For instance, in the VAS case, the reachability set can be finite and Ackermannian. But when it is infinite, there exist small witnesses of this fact [13]. This yields an optimal [11] exponential-space algorithm for the VAS boundedness problem. Extending this technique to PVAS is a challenging problem.

The boundedness problem for PVAS can be refined in two different ways. In fact, the infiniteness of the reachability set may come from the stack or the counters. We say that a PVAS is *counter-bounded* if the set of vectors  $\mathbf{c} \in \mathbb{N}^k$

such that  $(q, \mathbf{c}, w)$  is reachable for some  $q$  and  $w$ , is finite. Symmetrically, a PVAS is called *stack-bounded* if the set of words  $w \in \Gamma^*$  such that  $(q, \mathbf{c}, w)$  is reachable for some  $q$  and  $\mathbf{c}$ , is finite. The following lemma shows that the two associated decision problems are at least as hard as the boundedness problem.

**Lemma 1.2.** *The boundedness problem is reducible in logarithmic space to the counter-boundedness problem and to the stack-boundedness problem (the dimension  $k$  is unchanged by the reduction).*

The stack-boundedness problem can be solved by adapting the algorithm introduced in [8] for the PVAS boundedness problem. Informally, this algorithm explores the reachability tree and stops as soon as it detects a cycle of transitions whose iteration produces infinitely many reachable configurations. If this cycle increases the stack, we can immediately conclude stack-unboundedness. Otherwise, at least one counter can be increased to an arbitrary large number. By replacing the value of this counter by  $\omega$  and then resuming the computation of the tree from the new (extended) configuration, we obtain a Karp&Miller-like algorithm [5] deciding the stack-boundedness problem. We deduce the following result.

**Lemma 1.3.** *The stack-boundedness problem for PVAS is decidable.*

Concerning the counter-boundedness problem, adapting the algorithm introduced in [8] in a similar way seems to be more involved. Indeed, if we detect a cycle that only increases the stack, we can iterate it and represent its effect with a regular language. However, we do not know how to effectively truncate the resulting tree to obtain an algorithm deciding the counter-boundedness problem.

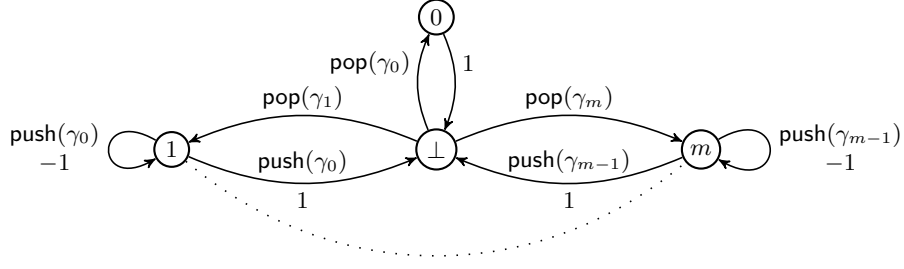
*Contributions.* In this paper we solve the counter-boundedness problem for the special case of dimension one. We show that in a grammar setting, PVAS counter-boundedness corresponds to the boundedness problem for prefix-closed, grammar-controlled vector addition systems. We show that in dimension one, this problem is decidable in exponential time. Our proof is based on the existence of small witnesses exhibiting the unboundedness property. This complexity result improves the best known upper bound for the classical boundedness problem for PVAS in dimension one. In fact, as shown by the following [Example 1.4](#), the reachability set of a bounded 1-dimensional PVAS can be Ackermannian large. In particular, the worst-case running time of the algorithm introduced in [8] for solving the boundedness problem is at least Ackermannian even in dimension one.

*Example 1.4.* The Ackermann functions  $A_m : \mathbb{N} \rightarrow \mathbb{N}$ , for  $m \in \mathbb{N}$ , are defined by induction for every  $n \in \mathbb{N}$  by:

$$A_m(n) \stackrel{\text{def}}{=} \begin{cases} n + 1 & \text{if } m = 0 \\ A_{m-1}^{n+1}(1) & \text{if } m > 0 \end{cases}$$

These functions are *weakly computable* by the (family of) PVAS depicted in [Figure 2](#), in the sense that:

$$A_m(n) = \max\{c \mid (\perp, n, \gamma_m) \xrightarrow{*} (\perp, c, \varepsilon)\} \quad (1)$$



**Fig. 2.** One-dimensional PVAS that weakly compute Ackermann functions.

for every  $m, n \in \mathbb{N}$ . Indeed, an immediate induction on  $k \in \{0, \dots, m\}$  shows that  $(\perp, c, \gamma_k) \xrightarrow{*} (\perp, A_k(c), \varepsilon)$  for every  $c \in \mathbb{N}$ . For the converse inequality, let us introduce, for each configuration  $(\perp, c, w)$ , the number  $\theta(c, w)$  defined by

$$\theta(c, \gamma_{i_1} \cdots \gamma_{i_k}) \stackrel{\text{def}}{=} A_{i_1} \circ \cdots \circ A_{i_k}(c)$$

An immediate induction on the number of times a run come back to the state  $\perp$  shows that  $(\perp, c, w) \xrightarrow{*} (\perp, c', w')$  implies  $\theta(c, w) \geq \theta(c', w')$ . Since  $\theta(c, \varepsilon) = c$ , we derive that  $A_m(n) \geq c$  for every  $c$  such that  $(\perp, n, \gamma_m) \xrightarrow{*} (\perp, c, \varepsilon)$ . This concludes the proof of [Equation \(1\)](#).

Notice that the reachability set of this PVAS is finite for any initial configuration. Indeed,  $(\perp, c, w) \xrightarrow{*} (\perp, c', w')$  implies  $\theta(c, w) \geq \theta(c', w') \geq c' + |w'|$ . Therefore, there are only finitely many reachable configurations in state  $\perp$ . It follows that the same property holds for the other states.  $\square$

*Outline.* We recall some necessary notations about context-free grammars and parse trees in the next section. In [Section 3](#), we present the model of grammar-controlled vector addition systems (GVAS) as previously introduced in [\[10\]](#), and reduce the counter boundedness problem for PVAS to the boundedness problem for the subclass of *prefix-closed* GVAS. We show in [Section 4](#) that unbounded systems exhibit certificates of a certain form. [Section 5](#) proves a technical lemma used later on and finally, in [Section 6](#), we bound the size of minimal certificates and derive the claimed exponential-time upper bound.

## 2 Preliminaries

We let  $\overline{\mathbb{Z}} \stackrel{\text{def}}{=} \mathbb{Z} \cup \{-\infty, +\infty\}$  denote the extended integers, and we use the standard extensions of  $+$  and  $\leq$  to  $\overline{\mathbb{Z}}$ . Recall that  $(\overline{\mathbb{Z}}, \leq)$  is a complete lattice.

*Words.* Let  $A^*$  be the set of all finite words over the alphabet  $A$ . The *empty word* is denoted by  $\varepsilon$ . We write  $|w|$  for the *length* of a word  $w$  in  $A^*$  and  $w^k \stackrel{\text{def}}{=} ww \cdots w$  for its  $k$ -fold concatenation. The *prefix* partial order  $\preceq$  over words is defined by

$u \preceq v$  if  $v = uw$  for some word  $w$ . We write  $u \prec v$  if  $u$  is a proper prefix of  $v$ . A language is a subset  $L \subseteq A^*$ . A language  $L$  is said to be *prefix-closed* if  $u \preceq v$  and  $v \in L$  implies  $u \in L$ .

*Trees.* A tree  $T$  is a finite, non-empty, prefix-closed subset of  $\mathbb{N}^*$  satisfying the property that if  $tj$  is in  $T$  then  $ti$  in  $T$  for all  $i < j$ . Elements of  $T$  are called *nodes*. Its *root* is the empty word  $\varepsilon$ . An *ancestor* of a node  $t$  is a prefix  $s \preceq t$ . A *child* of a node  $t$  in  $T$  is a node  $tj$  in  $T$  with  $j$  in  $\mathbb{N}$ . A node is called a *leaf* if it has no child (i.e.,  $t0 \notin T$ ), and is said to be *internal* otherwise. The *size* of a tree  $T$  is its cardinal  $|T|$ , its *height* is the maximal length  $|t|$  of its nodes  $t \in T$ . We let  $\preceq_{\text{lex}}$  denote the lexicographic order on words in  $\mathbb{N}^*$ .

*Context-free Grammars.* A *context-free grammar* is a quadruple  $G = (V, A, R, S)$ , where  $V$  and  $A$  are disjoint finite sets of *nonterminal* and *terminal* symbols,  $S \in V$  is a *start symbol*, and  $R \subseteq V \times (V \cup A)^*$  is a finite set of *production rules*. We write

$$X \vdash \alpha_1 \mid \alpha_2 \mid \dots \mid \alpha_k$$

to denote that  $(X, \alpha_1), \dots, (X, \alpha_k) \in R$ . For all words  $w, w' \in (V \cup A)^*$ , the grammar admits a *derivation step*  $w \Rightarrow w'$  if there exist two words  $u, v$  in  $(V \cup A)^*$  and a production rule  $(X, \alpha)$  in  $R$  such that  $w = uXv$  and  $w' = u\alpha v$ . Let  $\Rightarrow^*$  denote the reflexive and transitive closure of  $\Rightarrow$ . The *language* of a word  $w$  in  $(V \cup A)^*$  is the set  $L_w^G \stackrel{\text{def}}{=} \{z \in A^* \mid w \Rightarrow^* z\}$ . The *language* of  $G$  is defined as  $L_S^G$ , and it is denoted by  $L^G$ . A nonterminal  $X \in V$  is called *productive* if  $L_X^G \neq \emptyset$ . A context-free grammar  $G = (V, A, R, S)$  is in *Chomsky normal form*<sup>3</sup> if, for every production rule  $(X, \alpha)$  in  $R$ , either  $(X, \alpha) = (S, \varepsilon)$  or  $\alpha \in V^2 \cup A$ .

*Parse Trees.* A *parse tree* for a context-free grammar  $G = (V, A, R, S)$  is a tree  $T$  equipped with a labeling function  $\text{sym} : T \rightarrow (V \cup A \cup \{\varepsilon\})$  such that the root is labeled by  $\text{sym}(\varepsilon) = S$  and  $T$  contains the production rule  $\text{sym}(t) \vdash \text{sym}(t0) \dots \text{sym}(tk)$  for every internal node  $t$  with children  $t0, \dots, tk$ . In addition, each leaf  $t \neq \varepsilon$  with  $\text{sym}(t) = \varepsilon$  is the only child of its parent. Notice that  $\text{sym}(t) \in V$  for every internal node  $t$ . A parse tree is called *complete* when  $\text{sym}(t) \in (A \cup \{\varepsilon\})$  for every leaf  $t$ . The *yield* of a parse tree  $(T, \text{sym})$  is the word  $\text{sym}(t_1) \dots \text{sym}(t_\ell)$  where  $t_1, \dots, t_\ell$  are the leaves of  $T$  in lexicographic order (informally, from left to right). Observe that for every word  $w$  in  $(V \cup A)^*$ , it holds that  $S \Rightarrow^* w$  if, and only if,  $w$  is the yield of some parse tree.

### 3 Grammar-Controlled Vector Addition Systems

In this section we recall the notion of GVAS from [10] and show that the boundedness problem for the subclass of *prefix-closed* GVAS is inter-reducible to the counter-boundedness problem for pushdown vector addition systems.

<sup>3</sup> To simplify the presentation, we consider a weaker normal form than the classical one, as we allow to reuse the start symbol.

**Definition 3.1 (GVAS).** A  $k$ -dimensional grammar-controlled vector addition system (shortly, GVAS) is a tuple  $G = (V, \mathbf{A}, R, S, \mathbf{c}_{init})$  where  $(V, \mathbf{A}, R, S)$  is a context-free grammar,  $\mathbf{A} \subseteq \mathbb{Z}^k$  is a VAS, and  $\mathbf{c}_{init} \in \mathbb{N}^k$  is an initial vector.

The semantics of GVAS is given by extending the relations  $\xrightarrow{\mathbf{a}}$  of ordinary VAS to words over  $V \cup \mathbf{A}$  as follows. Define  $\xrightarrow{\varepsilon}$  to be the identity on  $\mathbb{N}^k$  and let  $\xrightarrow{z\mathbf{a}} \stackrel{\text{def}}{=} \xrightarrow{\mathbf{a}} \circ \xrightarrow{z}$  for  $z \in \mathbf{A}^*$  and  $\mathbf{a} \in \mathbf{A}$ . Finally, let  $\xrightarrow{w} \stackrel{\text{def}}{=} \bigcup_{z \in L_w^G} \xrightarrow{z}$  for  $w \in (V \cup \mathbf{A})^*$ . For a word  $z = \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n \in \mathbf{A}^*$  over the terminals, we shortly write  $\sum z$  for the sum  $\sum_{i=1}^n \mathbf{a}_i$ . Observe that  $\mathbf{c} \xrightarrow{z} \mathbf{d}$  implies  $\mathbf{d} - \mathbf{c} = \sum z$ .

Ultimately, we are interested in the relation  $\xrightarrow{S}$ , that describes the reachability relation via sequences of actions in  $L_S^G$ , i.e., those that are derivable from the starting symbol  $S$  in the underlying grammar. A vector  $\mathbf{d} \in \mathbb{N}^k$  is called *reachable* from a vector  $\mathbf{c} \in \mathbb{N}^k$  if  $\mathbf{c} \xrightarrow{S} \mathbf{d}$ . The *reachability set* of a GVAS is the set of vectors reachable from  $\mathbf{c}_{init}$ .

A GVAS is said to be *bounded* if its reachability set is finite. The associated boundedness problem for GVAS is challenging since the coverability problem for PVAS, whose decidability is still open, is logspace reducible to it. However, the various boundedness properties that we investigate on PVAS (see [Section 1](#)) consider all reachable configurations, without any acceptance condition. So they intrinsically correspond to context-free languages that are prefix-closed. It is therefore natural to consider the same restriction for GVAS. Formally, we call a GVAS  $G = (V, \mathbf{A}, R, S, \mathbf{c}_{init})$  *prefix-closed* when the language  $L_S^G$  is prefix-closed. Concerning the counter-boundedness problem for PVAS, the following lemma shows that it is sufficient to consider the special case of prefix-closed GVAS.

**Lemma 3.2.** *The counter-boundedness problem for PVAS is logspace inter-reducible with the prefix-closed GVAS boundedness problem (the dimension  $k$  is unchanged by both reductions).*

In this paper, we focus on the counter-boundedness problem for PVAS of dimension one. We show that this problem is decidable in exponential time. The proof is by reduction, using [Lemma 3.2](#), to the boundedness problem for prefix-closed 1-dimensional GVAS. Our main technical contribution is the following result.

**Theorem 3.3.** *The prefix-closed 1-dimensional GVAS boundedness problem is decidable in exponential time.*

For the remainder of the paper, we restrict our attention to the dimension one, and shortly write GVAS instead of 1-dimensional GVAS.

*Example 3.4.* Consider again the Ackermann functions  $A_m$  introduced in [Example 1.4](#). These can be expressed by the GVAS with nonterminals  $X_0, \dots, X_m$  and with production rules  $X_0 \vdash 1$  and  $X_i \vdash -1 X_i X_{i-1} \mid 1 X_{i-1}$  for  $1 \leq i \leq m$ . It is routinely checked that  $\max\{d \mid c \xrightarrow{X_m} d\} = A_m(c)$  for all  $c \in \mathbb{N}$ .  $\square$

Every GVAS can be effectively normalized, in logarithmic space, by replacing terminals  $a \in \mathbb{Z}$  by words over the alphabet  $\{-1, 0, 1\}$  and then putting the resulting grammar into Chomsky normal form. In addition, non-productive nonterminals, and production rules in which they occur, can be removed. So in order to simplify our proofs, we consider w.l.o.g. only GVAS of this simpler form.

*Assumption.* We restrict our attention to GVAS  $G = (V, A, R, S, c_{init})$  in Chomsky normal form and where  $A = \{-1, 0, 1\}$  and every  $X \in V$  is productive.

The rest of the paper is devoted to the proof of [Theorem 3.3](#). Before delving into its technical details, we give a high-level description of the proof. In the next section, we characterize unboundedness in terms of certificates, which are complete parse trees whose nodes are labeled by natural numbers (or  $-\infty$ ). These certificates contain a growing pattern that can be pumped to produce infinitely many reachable (1-dimensional) vectors, thereby witnessing unboundedness. We then prove that certificates need not be too large. To do so, we first show in [Section 5](#) how to bound the size of growing patterns. Then, we bound the height and labels of “minimal” certificates in [Section 6](#). Both bounds are singly-exponential in the size of the GVAS. Thus, the existence of a certificate can be checked by an alternating Turing machine running in polynomial space. This entails the desired EXPTIME upper-bound stated in [Theorem 3.3](#).

## 4 Certificates of Unboundedness

Following our previous work on the GVAS coverability problem [\[10\]](#), we annotate parse trees in a way that is consistent with the VAS semantics. A *flow tree* for a GVAS  $G = (V, A, R, S, c_{init})$  is a complete<sup>4</sup> parse tree  $(T, sym)$  for  $G$  equipped with two functions  $in, out : T \rightarrow \mathbb{N} \cup \{-\infty\}$ , assigning an *input* and an *output* value to each node, with  $in(\varepsilon) = c_{init}$ , and satisfying, for every node  $t \in T$ , the following *flow conditions*:

1. If  $t$  is internal with children  $t0, \dots, tk$ , then  $in(t0) \leq in(t)$ ,  $out(t) \leq out(tk)$ , and  $in(t(j+1)) \leq out(tj)$  for every  $j = 0, \dots, k-1$ .
2. If  $t$  is a leaf, then  $out(t) \leq in(t) + a$  if  $sym(t) = a \in A$ , and  $out(t) \leq in(t)$  if  $sym(t) = \varepsilon$ .

We shortly write  $t : c \# d$  to mean that  $(in(t), sym(t), out(t)) = (c, \#, d)$ . The *size* of a flow tree is the size of its underlying parse tree. [Figure 3](#) (left) shows a flow tree for the GVAS of [Example 3.4](#), with start symbol  $X_1$  and initial (1-dimensional) vector  $c_{init} = 5$ .

*Remark 4.1.* The flow conditions enforce the VAS semantics along a depth-first pre-order traversal of the complete parse tree. But, as in [\[10\]](#), we only require inequalities instead of equalities. This corresponds to a *lossy* VAS semantics, where

<sup>4</sup> Compared to [\[10\]](#) where flow trees are built on arbitrary parse trees, the flow trees that we consider here are always built on complete parse trees.



the counter can be non-deterministically decreased [2]. The use of inequalities in our flow conditions simplifies the presentation and allows for certificates of unboundedness with smaller input/output values. Note that equalities would be required to get certificates of reachability, but the latter problem is out of the scope of this paper.

**Lemma 4.2.** *For all  $d$  with  $c_{init} \xrightarrow{S} d$ , there exists a flow tree with  $out(\varepsilon) = d$ .*

Our main ingredient to prove Theorem 3.3 is a small model property. First, we show in this section that unboundedness can always be witnessed by a flow tree of a particular form, called a certificate (see Definition 4.5 and Figure 3). Then, we will provide in Theorem 6.10 exponential bounds on the height and input/output values of “minimal” certificates. This will entail the desired EXPTIME upper-bound for the prefix-closed GVAS boundedness problem.

We start by bounding the size of flow trees that do not contain an iterable pattern, i.e., a nonterminal that repeats, below it, with a larger or equal input value. Formally, a flow tree  $(T, sym, in, out)$  is called *good* if it contains a node  $t$  and a proper ancestor  $s \prec t$  such that  $sym(s) = sym(t)$  and  $in(s) \leq in(t)$ . It is called *bad* otherwise. We bound the size of bad flow trees by (a) translating them into bad nested sequences, and (b) using a bound given in [8] on the length of bad nested sequences. Let us first recall some notions and results from [8]. Our presentation is deliberately simplified and limited to our setting.

Let  $(S, \preceq, \|\cdot\|)$  be the normed quasi-ordered set defined by  $S \stackrel{\text{def}}{=} V \times \mathbb{N}$ ,  $(X, m) \preceq (Y, n) \stackrel{\text{def}}{\iff} X = Y \wedge m \leq n$ , and  $\|(X, m)\| = m$ . A *nested sequence* is a finite sequence  $(s_1, h_1), \dots, (s_\ell, h_\ell)$  of elements in  $S \times \mathbb{N}$  satisfying  $h_1 = 0$  and  $h_{j+1} \in h_j + \{-1, 0, 1\}$  for every index  $j < \ell$  of the sequence. A nested sequence  $(s_1, h_1), \dots, (s_\ell, h_\ell)$  is called *good* if there exists  $i < j$  such that  $s_i \preceq s_j$  and  $h_i \leq h_{i+1}, \dots, h_j$ . A *bad* nested sequence is one that is not good. A nested sequence  $(s_1, h_1), \dots, (s_\ell, h_\ell)$  is called *n-controlled*, where  $n \in \mathbb{N}$ , if  $\|s_j\| < n + j$  for every index  $j$  of the sequence.

**Theorem 4.3 ([8, Theorem VI.1]).** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ . Every  $n$ -controlled bad nested sequence has length at most  $F_{\omega, |V|}(n)$ .*

The function  $F_{\omega, |V|} : \mathbb{N} \rightarrow \mathbb{N}$  used in the theorem is part of the *fast-growing hierarchy*. Its precise definition (see, e.g., [8]) is not important for the rest of the paper. The following lemma provides a bound on the size of bad flow trees. Notice that this lemma applies to arbitrary GVAS (not necessarily prefix-closed).

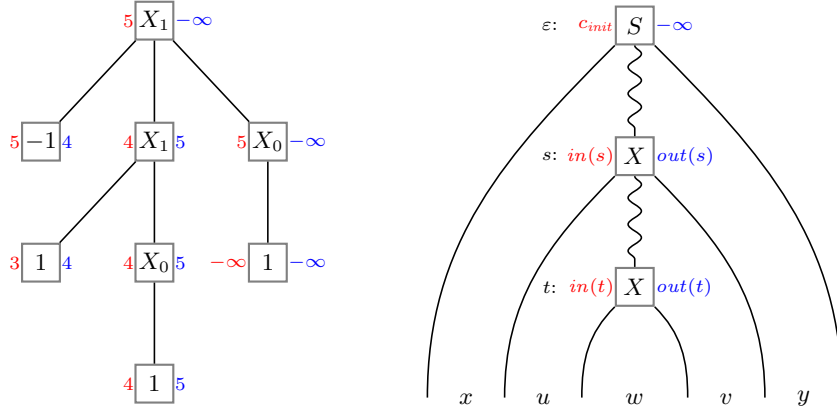
**Lemma 4.4.** *Every bad flow tree has at most  $F_{\omega, |V|}(c_{init} + 2)$  nodes.*

A good flow tree contains an iterable pattern that can be “pumped”. However, the existence of such a pattern does not guarantee unboundedness. For that, we need stronger requirements on the input and output values, as defined below.

**Definition 4.5 (Certificates).** *A certificate for a given GVAS is a flow tree  $(T, sym, in, out)$  equipped with two nodes  $s \prec t$  in  $T$  such that*

$$sym(s) = sym(t) \quad \text{and} \quad in(s) \leq in(t) \quad \text{and} \quad in(s) < in(t) \text{ or } out(t) < out(s)$$





**Fig. 3. Left:** a flow tree for the GVAS of [Example 3.4](#) with  $c_{init} = 5$ . Input and output values are indicated in red and blue, respectively. **Right:** A certificate with  $sym(t) = sym(s) = X$  and yield  $xuwvy \in A^*$ . It must hold that either  $in(s) < in(t)$  or  $in(s) = in(t)$  and  $out(t) < out(s)$ .

We now present the main result of this section, which shows that unboundedness can always be witnessed by a certificate.

**Theorem 4.6.** *A prefix-closed GVAS  $G$  is unbounded if, and only if, there exists a certificate for  $G$ .*

## 5 Growing Patterns

Certificates depicted on [Figure 3](#) (right) introduce words  $u \in A^*$  satisfying a sign constraint  $\sum u > 0$  or  $\sum u = 0$ . These words are derivable from words of non-terminal symbols  $S_1 \dots S_k$  corresponding to the left children of the nodes between  $s$  and  $t$ . In order to obtain small certificates, in this section, we provide bounds on the minimal length of words  $u' \in A^*$  that can also be derived from  $S_1 \dots S_k$  and that satisfy the same sign constraint as  $u$ .

Let us first introduce the *displacement* of a GVAS  $G$  as the “best shift” achievable by a word in  $L^G$  and defined by the following equality<sup>5</sup>:

$$\Delta^G \stackrel{\text{def}}{=} \sup\{\sum z \mid z \in L^G\}$$

When the displacement is finite, the following [Lemma 5.1](#) shows that it is achievable by a complete elementary parse tree. We say that a parse tree  $T$  is *elementary* if for every  $s \preceq t$  such that  $sym(s) = sym(t)$ , we have  $s = t$ . Notice that the size of an elementary parse tree is bounded by  $2^{|V|+1}$ .

**Lemma 5.1.** *Every GVAS  $G$  admits a complete elementary parse with a yield  $w$  such that  $\Delta^G \in \{\sum w, +\infty\}$ .*

<sup>5</sup> Notice that  $\Delta^G$  may be negative.

Given a non-terminal symbol  $X$ , we denote by  $G[X]$  the context-free grammar obtained from  $G$  by replacing the start symbol by  $X$ . We are now ready to state the main observation of this section.

**Theorem 5.2.** *For every sequence  $S_1, \dots, S_k$  of non-terminal symbols of a GVAS  $G$  there exists a sequence  $T_1, \dots, T_k$  of complete parse trees for  $G_j \stackrel{\text{def}}{=} G[S_j]$  with a yield  $z_j$  such that  $|T_1| + \dots + |T_k| \leq 3k4^{|V|+1}$ , and such that  $\sum z_1 \dots z_k > 0$  if  $\Delta^{G_1} + \dots + \Delta^{G_k} > 0$ , and  $\sum z_1 \dots z_k = 0$  if  $\Delta^{G_1} + \dots + \Delta^{G_k} = 0$ .*

We first provide bounds on complete parse trees that witness the following properties  $\Delta^G = +\infty$  and  $X$  is derivable. Formally, a nonterminal  $X$  is said to be *derivable* if there exists  $w \in (A \cup V)^*$  that contains  $X$  and such that  $S \xRightarrow{*} w$ .

**Lemma 5.3.** *If  $\Delta^G = +\infty$ , there exists a parse tree for  $G[X]$  where  $X$  is a non-terminal symbol derivable from the start symbol  $S$  with a yield  $uXv$  satisfying  $u, v \in A^*$ ,  $\sum uv > 0$ , and a number of nodes bounded by  $4^{|V|+1}$ .*

**Lemma 5.4.** *For every derivable non-terminal symbol  $X$ , there exists a parse tree with a yield in  $A^*XA^*$  and a number of nodes bounded by  $4^{|V|+1}$ .*

*Proof (of Theorem 5.2).* We can assume that  $k \geq 1$  since otherwise the proof is trivial. Observe that if  $\Delta^{G_1} + \dots + \Delta^{G_k} < +\infty$  then  $\Delta^{G_j} < +\infty$  for every  $j$ . It follows from Lemma 5.1 that there exists a complete parse tree  $T_j$  for  $G[S_j]$  with a yield  $w_j$  satisfying  $\Delta^{G_j} = \sum w_j$  and a number of nodes bounded by  $2^{|V|+1}$ . Thus  $|T_1| + \dots + |T_k| \leq k2^{|V|+1}$  and  $\sum w_1 \dots w_k = \Delta^{G_1} + \dots + \Delta^{G_k}$ . So, in this special case the theorem is proved. Now, let us assume that  $\Delta^{G_1} + \dots + \Delta^{G_k} = +\infty$ . There exists  $p \in \{1, \dots, k\}$  such that  $\Delta^{G_p} = +\infty$ . Lemma 5.3 shows that there exists a variable for  $X$  derivable from  $S_p$  and a parse tree  $T_+$  for  $G[X]$  with a yield  $uXv$  satisfying  $u, v \in A^*$ ,  $\sum uv > 0$ , and such that  $|T_+| \leq 4^{|V|+1}$ . Since  $S_j$  is productive, there exists a complete elementary parse tree  $T_j$  for  $G[S_j]$  with a yield  $w_j \in A^*$ . For the same reason, there exists a complete elementary parse tree  $T$  for  $G[X]$  with a yield  $w \in A^*$ . As  $X$  is derivable from  $S$ , Lemma 5.4 shows that there exists a parse tree  $T'$  for  $G$  with a yield labeled by a word in  $u'Xv'$  with  $u', v' \in A^*$ , and a number of nodes bounded by  $4^{|V|+1}$ . Notice that for any  $n \in \mathbb{N}$ , we deduce a complete parse tree  $T_p$  for  $G[S_p]$  with a yield  $w_p = u'u^n w v^n v'$  by inserting in  $T'$  many ( $n$ ) copies of  $T_+$  and one copy of  $T$ . Observe that  $\sum w_1 \dots w_k \geq -|w_1 \dots w_{p-1} w_{p+1} \dots w_k| - |u' w v'| + n \sum uv \geq -k2^{|V|+1} - 4^{|V|+1} + n$ . Let us fix  $n$  to  $2k4^{|V|+1} - 2$ . It follows that  $\sum w_1 \dots w_p > 0$ . Moreover, we have  $|T_p| \leq |T| - 1 + n(|T_+| - 1) + |T'| \leq 2^{|V|+1} + n4^{|V|+1} + 4^{|V|+1} \leq (n+2)4^{|V|+1} \leq 2k4^{|V|+1}$ . We derive  $|T_1| + \dots + |T_k| \leq (k-1)2^{|V|+1} + 2k4^{|V|+1} \leq 3k4^{|V|+1}$ . We have proved Theorem 5.2.  $\square$

## 6 Small Certificates

We provide in this section exponential bounds on the height and input/output values of minimal certificates in the following sense. Let the *rank* of a flow tree

$(T, \text{sym}, \text{in}, \text{out})$  be the pair

$$\left( |T_{\text{in}}| + |T_{\text{out}}|, \sum_{t \in T_{\text{in}}} \text{in}(t) + \sum_{t \in T_{\text{out}}} \text{out}(t) \right)$$

where  $T_{\text{in}} = \{t \in T \mid \text{in}(t) > -\infty\}$  and  $T_{\text{out}} = \{t \in T \mid \text{out}(t) > -\infty\}$ . Notice that  $T_{\text{out}} \subseteq T_{\text{in}}$ . We compare ranks using the lexicographic order  $\preceq_{\text{lex}}$  over  $\mathbb{N}^2$  and let the rank of a certificate  $(\mathcal{T}, s, t)$  be the rank of its flow tree  $\mathcal{T}$ .

Consider a prefix-closed GVAS  $G = (V, A, R, S, c_{\text{init}})$  that is unbounded. By [Theorem 4.6](#), there exists a certificate for  $G$ . Pick a certificate  $(\mathcal{T}, s, t)$  among those of least rank. Our goal is to bound the height and input/output values of  $\mathcal{T}$ . Based on its assumed minimality, we observe a series of facts about our chosen certificate.

First, we observe that some input/output values in  $\mathcal{T}$  must be  $-\infty$ , because higher values would be useless in the sense that they can be set to  $-\infty$  without breaking the flow conditions nor the conditions on  $s$  and  $t$ . This observation is formalized in the two following facts.

*Fact 6.1.* It holds that  $\text{out}(p) = -\infty$  for every proper ancestor  $p \prec s$ . Moreover,  $\text{in}(p) = \text{out}(p) = -\infty$  for every node  $p \in T$  such that  $s \prec_{\text{lex}} p$  and  $p \not\preceq s$ .

*Fact 6.2.* Assume that  $\text{in}(s) < \text{in}(t)$ . It holds that  $\text{out}(p) = -\infty$  for every ancestor  $p \preceq t$ . Moreover,  $\text{in}(p) = \text{out}(p) = -\infty$  for all  $p \in T$  with  $t \prec_{\text{lex}} p$ .

Next, we observe that the main branch, that contains  $s$  and  $t$ , must be short.

*Fact 6.3.* It holds that  $|s| \leq |V|$  and  $|t| \leq |s| + |V| + 1$ .

The next two facts provide *relative* bounds on input and output values for nodes that are not on the main branch.

*Fact 6.4.* It holds that  $\text{in}(p) \leq \text{out}(p) + 2^{|V|}$  for every node  $p \in T$  with  $p \not\preceq t$ .

*Fact 6.5.* Let  $q \in T$  and let  $p$  be the parent of  $q$ . If  $p = t$  or  $p \not\preceq t$ , then  $\text{out}(q) \leq \text{out}(p) + 2^{|V|}$ . If moreover  $\text{sym}(p) = \text{sym}(q)$ , then  $\text{out}(q) < \text{out}(p)$ .

The following facts provide *absolute* bounds on the input/output values of nodes  $s$  and  $t$ . The proofs of the facts below crucially rely on [Section 5](#). Consider the subtrees on the left and on the right of the branch from  $s$  to  $t$ . The main idea of the proofs is to replace these subtrees by small ones using [Theorem 5.2](#).

*Fact 6.6.* It holds that  $\text{out}(t) \leq \text{out}(s) \leq 6|V| \cdot 4^{|V|+1}$ .

*Fact 6.7.* It holds that  $\text{in}(s) \leq \text{in}(t) \leq 7|V| \cdot 4^{|V|+1}$ .

Now we derive absolute bounds for the input/output values of the remaining nodes on the main branch. These are derived from [Facts 6.6](#) and [6.7](#), using [Facts 6.4](#) and [6.5](#) about the way in/output values propagate and the [Fact 6.3](#) that the intermediate path between nodes  $s$  and  $t$  is short.

*Fact 6.8.* It holds that  $out(p) \leq 4^{2(|V|+1)}$  for every ancestor  $p \preceq t$ .

*Fact 6.9.* It holds that  $in(p) \leq 4^{2(|V|+1)}$  for every ancestor  $\varepsilon \prec p \preceq t$ .

We are now ready to derive bounds on the rank of our minimal certificate. Notice that it remains only to bound the depth and the input/output values on branches different from the main branch.

Consider therefore a node  $q$  outside the main branch, i.e.,  $q \not\preceq t$ . Let  $p$  be the least prefix of  $q$  such that  $p = t$  or  $p \not\preceq t$ . We first show that  $out(p) \leq 4^{2(|V|+1)}$ . If  $p = t$  then the claim follows from [Fact 6.8](#). Otherwise, the parent  $r$  of  $p$  satisfies  $r \prec t$ . Observe that the other child  $\bar{p}$  of  $r$  satisfies  $\bar{p} \preceq t$ . The flow conditions together with the minimality of  $(\mathcal{T}, s, t)$  guarantee that

- if  $p = r1$  then  $out(p) = out(r)$ , hence,  $out(p) \leq 4^{2(|V|+1)}$  by [Fact 6.8](#), and
- if  $p = r0$  then  $out(p) = in(r1)$ , hence,  $out(p) \leq 4^{2(|V|+1)}$  by [Fact 6.9](#).

According to [Fact 6.5](#), the output values on the branch from  $p$  down to  $q$  may only increase when visiting a new symbol. Moreover, this increase is bounded by  $2^{|V|}$ . It follows that  $out(r) \leq out(p) + |V|2^{|V|}$  for every node  $r$  such that  $p \prec r \preceq q$ . [Fact 6.4](#) entails that  $in(r) \leq out(p) + (|V| + 1)2^{|V|}$ . We obtain that  $\max\{in(r), out(r)\} < 4^{3(|V|+1)}$  for every node  $r$  with  $p \prec r \preceq q$ . [Fact 6.5](#) also forbids the same nonterminal from appearing twice with the same output value, so  $|r| \leq |p| + |V| \cdot 4^{3(|V|+1)} + 1$ . Observe that  $|p| \leq |t|$ . We derive from [Fact 6.3](#) that  $|r| \leq 4^{4(|V|+1)}$ . This concludes the proof of the following theorem.

**Theorem 6.10.** *A prefix-closed GVAS  $(V, A, R, S, c_{init})$  is unbounded if, and only if, it admits a certificate with height and all input/output values bounded by  $c_{init} + 4^{4(|V|+1)}$ .*

*Proof (of [Theorem 3.3](#)).* By [Theorem 6.10](#), a certificate for unboundedness is a flow tree of exponential height and with all input and output labels exponentially bounded. An alternating Turing machine can thus guess and verify all branches of such a flow tree, storing intermediate input/output values as well as the remaining length of a branch in polynomial space. The claim then follows from the fact that alternating polynomial space equals exponential time.  $\square$

## 7 Conclusion

We discussed different boundedness problems for pushdown vector addition systems [\[8, 7\]](#), which are a known, and very expressive computational model that features nondeterminism, a pushdown stack and several counters. These systems

may be equivalently interpreted, in the context of regulated rewriting [3], as vector addition systems with context-free control languages.

We observe that boundedness is reducible to both counter- and stack-boundedness. The stack boundedness problem can be shown to be decidable (with hyper-Ackermannian complexity) by adjusting the algorithm presented in [8].

Here, we single out the special case of the counter-boundedness problem for one-dimensional systems and propose an exponential-time algorithm that solves it. This also improves the best previously known Ackermannian upper bound for boundedness in dimension one.

Currently, the best lower bound for this problem is NP, which can be seen by reduction from the subset sum problem. For dimension two, PSPACE-hardness follows by reduction from the state-reachability of bounded one-counter automata with succinct counter updates [4]. For arbitrary dimensions, TOWER-hardness is known already for the boundedness problem [7,8] but the decidability of counter-boundedness for PVAS remains open.

### Acknowledgments

The authors wish to thank M. Praveen for insightful discussions. We also thank the anonymous referees for their useful comments and suggestions.

### References

1. Ball, T., Majumdar, R., Millstein, T.D., Rajamani, S.K.: Automatic predicate abstraction of C programs. In: PLDI. pp. 203–213 (2001)
2. Bouajjani, A., Mayr, R.: Model checking lossy vector addition systems. In: STACS. pp. 323–333 (1999)
3. Dassow, J., Pun, G., Salomaa, A.: Grammars with controlled derivations. In: Handbook of Formal Languages, pp. 101–154 (1997)
4. Fearnley, J., Jurdzinski, M.: Reachability in two-clock timed automata is pspace-complete. In: ICALP. pp. 212–223 (2013)
5. Karp, R.M., Miller, R.E.: Parallel program schemata. J. Comput. Syst. Sci. 3(2), 147–195 (1969)
6. Kosaraju, S.R.: Decidability of reachability in vector addition systems (preliminary version). In: STOC. pp. 267–281 (1982)
7. Lazic, R.: The reachability problem for vector addition systems with a stack is not elementary. CoRR abs/1310.1767 (2013)
8. Leroux, J., Praveen, M., Sutre, G.: Hyper-ackermannian bounds for pushdown vector addition systems. In: CSL/LICS (2014)
9. Leroux, J., Schmitz, S.: Demystifying reachability in vector addition systems. In: LICS (2015)
10. Leroux, J., Sutre, G., Totzke, P.: On the coverability problem for pushdown vector addition systems in one dimension. In: ICALP. pp. 324–336 (2015)
11. Lipton, R.J.: The reachability problem requires exponential space. Tech. Rep. 63, Yale University (Jan 1976)
12. Mayr, E.W.: An algorithm for the general Petri net reachability problem. In: STOC. pp. 238–246 (1981)
13. Rackoff, C.: The covering and boundedness problems for vector addition systems. TCS 6(2), 223–231 (1978)

## A Missing Proofs

### A.1 Proofs for Section 1

**Lemma 1.2.** *The boundedness problem is reducible in logarithmic space to the counter-boundedness problem and to the stack-boundedness problem (the dimension  $k$  is unchanged by the reduction).*

*Proof.* Let us consider a PVAS  $A$  and let us introduce a PVAS  $A'$  such that  $A$  is bounded if, and only if,  $A'$  is bounded, and such that if  $A$  is unbounded then  $A'$  is both counter-unbounded and stack-unbounded. The system  $A'$  is a copy of  $A$ , extended with a new state  $\perp$ . The state  $\perp$  has self-loops that allow to pop any symbol from the stack and simultaneously increment the first counter. It also has self-loops that decrease any counter and push some symbol to the stack. Finally, we add a transition  $(q, \mathbf{0}, \text{nop}, \perp)$  for each original state  $q$ . Now just observe that  $A$  is bounded if, and only if,  $A'$  is bounded. Moreover if  $A$  is unbounded, then  $A'$  is both counter-unbounded and stack-unbounded.  $\square$

### A.3 Proofs for Section 3

**Lemma 3.2.** *The counter-boundedness problem for PVAS is logspace inter-reducible with the prefix-closed GVAS boundedness problem (the dimension  $k$  is unchanged by both reductions).*

*Proof.* Just observe that a PVAS can be interpreted as a pushdown automaton that recognizes a context-free and prefix-closed trace language  $L \subseteq \mathbf{A}^*$  where  $\mathbf{A} \subseteq \mathbb{Z}$  is the set of vectors labeling the transitions. We can construct, in logarithmic space, a context-free grammar that produces  $L$ . This context-free grammar (equipped with the initial value  $\mathbf{c}_{init}$  of the PVAS) is a prefix-closed GVAS  $G$ . The reachability set of  $G$  is exactly the set of vectors  $\mathbf{c}$  such that  $(q, \mathbf{c}, w)$  is a reachable configuration of the PVAS for some  $q$  and  $w$ . The converse construction follows a similar idea by observing that the language of a prefix-closed context-free grammar can be accepted by a pushdown automaton (with all states accepting), computable in logarithmic space.  $\square$

**Lemma A.1.** *Let  $G = (V, A, R, S, \mathbf{c}_{init})$  be a GVAS with  $\max\{|\mathbf{a}| : \mathbf{a} \in A\} \leq n$ . One can construct, in logspace, an equivalent GVAS  $G' = (V', A', R', S', \mathbf{c}_{init})$  with the same reachability set and such that  $A' = \{-1, 0, 1\}$ .*

*Proof.*  $G'$  will be a copy of  $G$ , extended as follows. For all  $1 \leq m \leq n$ , there are new nonterminals  $B_m$  and rules  $B_1 \vdash 1$  and  $B_m \vdash B_{m-1}B_{m-1}$  for  $m > 0$ . Now all terminals  $\mathbf{a} \in \mathbf{A}$  such that  $\mathbf{a} > 1$  are removed from  $A'$  and replaced (on right hand sides of all rules) by a new nonterminal  $X_a$ . The only rule that rewrites this symbol is

$$X_a \vdash B_n^{b_n} B_{n-1}^{b_{n-1}} \cdots B_1^{b_1} \quad (2)$$

where  $a = b_n b_{n-1} \dots b_1$  is the binary representation of  $a$ . Note that  $B_m^{b_m}$  is the empty word if  $b_m = 0$  and  $B_m$  otherwise. We thus observe that the language  $L_{X_a}^{G'}$

contains only the word  $1^a$ . In particular, for any  $c, d \in \mathbb{N}$  we get that  $c \xrightarrow{a} d$  iff  $c \xrightarrow{X_a} d$ .

An analogous construction allows to replace all terminals  $a \in \mathbf{A}$  with  $a < 1$ . The resulting GVAS  $G'$  has terminal alphabet  $A' = \{-1, 0, 1\}$  as required.  $\square$

#### A.4 Proofs for Section 4

**Lemma 4.2.** *For all  $d$  with  $c_{init} \xrightarrow{S} d$ , there exists a flow tree with  $\text{out}(\varepsilon) = d$ .*

*Proof.* Assume that  $c_{init} \xrightarrow{S} d$ . It holds that  $c_{init} \xrightarrow{z} d$  for some  $z \in L_S$ . Since  $z \in L_S$ , there exists a derivation  $S \xRightarrow{*} z$ , hence, a complete parse tree with root labeled by  $S$  and yield  $z$ . This complete parse tree, together with the fact that  $c_{init} \xrightarrow{z} d$ , induces a flow tree with root  $\varepsilon : c_{init} S d$ .  $\square$

**Lemma 4.4.** *Every bad flow tree has at most  $F_{\omega, |V|}(c_{init} + 2)$  nodes.*

*Proof.* Let  $\mathcal{T} = (T, \text{sym}, \text{in}, \text{out})$  be a flow tree. We construct a nested sequence that corresponds to a depth-first pre-order traversal of the flow tree. Let us introduce, for each symbol  $X \in V$ , two copies  $X'$  and  $X''$ . We associate to each node  $t \in \mathbb{N}^*$  a word  $\theta(t)$  over  $S \times \mathbb{N}$ , inductively defined as follows:

$$\theta(t) = \begin{cases} ((X, m), |t|) \cdot \theta(t0) \cdot ((X', m), |t|) \cdot \theta(t1) \cdot ((X'', m), |t|) & \text{if } t0 \in T \\ \varepsilon & \text{otherwise} \end{cases}$$

where  $X = \text{sym}(t)$  and  $m = \text{in}(t)$ . Recall that the condition  $t0 \in T$  means that  $t$  is an internal node of  $T$ . It is readily seen that  $\theta(\varepsilon)$  is a nested sequence. Let us write it as  $\theta(\varepsilon) = (s_1, h_1), \dots, (s_\ell, h_\ell)$ . Obviously, every index  $1 \leq i \leq \ell$  can be mapped back to a node  $t(i)$  of the flow tree  $\mathcal{T}$ .

Assume that  $\mathcal{T}$  is bad, and suppose, towards a contradiction, that  $\theta(\varepsilon)$  is good. So there exists  $i < j$  such that  $s_i \preceq s_j$  and  $h_i \leq h_{i+1}, \dots, h_j$ . This entails that  $t(i)$  is an ancestor of  $t(j)$ . Moreover,  $t(i) \neq t(j)$  because each node  $t \in T$  is visited three times in the sequence  $\theta(\varepsilon)$ , and each visit uses a different copy of  $\text{sym}(t)$ . So  $t(i)$  is a proper ancestor of  $t(j)$ . Since  $s_i \preceq s_j$ , we get that  $\text{sym}(t(i)) = \text{sym}(t(j))$  and  $\text{in}(t(i)) \leq \text{in}(t(j))$ , which contradicts our assumption that  $\mathcal{T}$  is bad.

We have shown that the nested sequence  $\theta(\varepsilon)$  is bad. Let us show that  $\theta(\varepsilon)$  is  $c_{init}$ -controlled. Let  $1 \leq j \leq \ell$ . Since  $A = \{-1, 0, 1\}$  by assumption, it holds that  $\text{in}(t(j)) \leq c_{init} + L$  where  $L$  denotes the number of leaves that are lexicographically smaller than  $t(j)$ . Recall that  $G$  is in Chomsky normal form by assumption. So these leaves have distinct parents, and those are all visited before  $t(j)$  in the sequence  $\theta(\varepsilon)$ , hence,  $j > L$ . It follows that  $\|s_j\| = \text{in}(t(j)) < c_{init} + j$ .

Since the nested sequence  $\theta(\varepsilon)$  is  $(c_{init} + 2)$ -controlled and bad, we derive from Theorem 4.3 that its length  $\ell$  satisfies  $\ell \leq F_{\omega, |V|}(c_{init} + 2)$ . The observation that  $|T| \leq \ell$  concludes the proof.  $\square$

**Theorem 4.6.** *A prefix-closed GVAS  $G$  is unbounded if, and only if, there exists a certificate for  $G$ .*



*Proof.* Assume that the reachability set of  $G$  is infinite. So there exists  $d$  such that  $c_{init} \xrightarrow{S} d$  and  $d > c_{init} + F_{\omega, |V|}(c_{init} + 2)$ . Pick a flow tree  $\mathcal{T} = (T, \text{sym}, \text{in}, \text{out})$  with root  $\varepsilon : c_{init} S d$ , among those of least size. Note that such a flow tree exists by Lemma 4.2. Let  $z \in A^*$  denote the yield of the complete parse tree  $(T, \text{sym})$ . It is readily seen that  $c_{init} \xrightarrow{z} e$  for some  $e \geq d$ . Recall that  $A = \{-1, 0, 1\}$  by assumption. It follows that  $|z| \geq \sum z = e - c_{init} > F_{\omega, |V|}(c_{init} + 2)$ . Observe that  $|T| \geq |z|$  since  $z$  is the yield of  $T$ . It follows from Lemma 4.4 that the flow tree  $(T, \text{sym}, \text{in}, \text{out})$  is good. So it contains a node  $t$  and a proper ancestor  $s \prec t$  such that  $\text{sym}(s) = \text{sym}(t)$  and  $\text{in}(s) \leq \text{in}(t)$ . To prove that  $(\mathcal{T}, s, t)$  is a certificate for  $G$ , it suffices to show that  $\text{in}(s) < \text{in}(t)$  or  $\text{out}(t) < \text{out}(s)$ . Assume, by contradiction, that  $\text{in}(s) = \text{in}(t)$  and  $\text{out}(t) \geq \text{out}(s)$ . We may replace, without breaking the flow conditions, the subtree rooted in  $s$  by the subtree rooted in  $t$ . We may even preserve the input and output of  $s$ . The resulting flow tree also has root  $\varepsilon : c_{init} S d$ , but it has less nodes than  $\mathcal{T}$ , which contradicts the minimality of  $\mathcal{T}$ .

Conversely, assume that there exists a certificate  $(\mathcal{T}, s, t)$  for  $G$ , with  $\mathcal{T} = (T, \text{sym}, \text{in}, \text{out})$ . By definition, it holds that  $s \prec t$ ,  $\text{sym}(s) = \text{sym}(t)$ , and either  $\text{in}(s) < \text{in}(t)$  or  $\text{in}(s) = \text{in}(t)$  and  $\text{out}(t) < \text{out}(s)$ . Let  $X$  denote the common nonterminal  $X = \text{sym}(s) = \text{sym}(t)$ . We decompose the yield  $z \in A^*$  of the complete parse tree  $(T, \text{sym})$  into  $z = xuwvy$ , as depicted in Figure 3, where:

- $x$  and  $y$  come from the leaves that are lexicographically smaller and larger than  $s$ , respectively,
- $u$  and  $v$  come from the leaves of the subtree rooted in  $s$  that are lexicographically smaller and larger than  $t$ , respectively, and
- $w$  comes from the leaves of the subtree rooted in  $t$ .

It is readily seen that  $S \xRightarrow{*} xXy$ ,  $X \xRightarrow{*} uXv$  and  $X \xRightarrow{*} w$ . Since  $L_S$  is prefix-closed, we get that  $\{xu^n \mid n \in \mathbb{N}\} \subseteq L_S$  and  $\{xu^n wv^n \mid n \in \mathbb{N}\} \subseteq L_S$ . Observe that  $\text{in}(t) \leq \text{in}(s) + \sum u$  and  $\text{out}(s) \leq \text{out}(t) + \sum v$ . These two inequalities follow from the flow conditions. There are two cases.

1. Either  $\text{in}(s) < \text{in}(t)$ , in which case  $\sum u > 0$ . Since  $-\infty < \text{in}(t)$ , it holds that  $c_{init} \xrightarrow{xu} d$  for some  $d$ . We derive that  $c_{init} \xrightarrow{x} c \xrightarrow{u^n} c + n \cdot \sum u$  for every  $n \in \mathbb{N}$ , where  $c = c_{init} + \sum x$ . Since  $\sum u > 0$ , we derive that  $\{d \mid c_{init} \xrightarrow{S} d\}$  is infinite.
2. Or  $\text{in}(s) = \text{in}(t)$  and  $\text{out}(t) < \text{out}(s)$ , in which case  $\sum u \geq 0$  and  $\sum v > 0$ . Since  $-\infty < \text{out}(s)$ , it holds that  $c_{init} \xrightarrow{xuwv} d$  for some  $d$ . We derive that  $c_{init} \xrightarrow{xu^n w} c + n \cdot \sum u \xrightarrow{v^n} c + n \cdot \sum uv$  for every  $n \in \mathbb{N}$ , where  $c = c_{init} + \sum xw$ . Since  $\sum uv > 0$ , we derive that  $\{d \mid c_{init} \xrightarrow{S} d\}$  is infinite.

In both cases, we obtain that the reachability set of  $G$  is infinite.  $\square$

## A.5 Proofs for Section 5

**Lemma 5.1.** *Every GVAS  $G$  admits a complete elementary parse with a yield  $w$  such that  $\Delta^G \in \{\sum w, +\infty\}$ .*

*Proof.* Notice that if  $\Delta^G < +\infty$  there exists a complete parse tree with a yield  $w$  such that  $\sum w = \Delta^G$ . Since  $G$  admits a complete parse tree, we can pick a complete parse tree  $T$  that is minimal wrt. the number of nodes and with a yield  $w$  that satisfies  $\Delta^G \in \{\sum w, +\infty\}$ . Assume by contradiction that  $T$  is not elementary. In that case, there exist two distinct nodes  $s \prec t$  that are labeled by the same non-terminal symbol  $X \in V$ . Notice that if  $\Delta^G = +\infty$ , by collapsing in  $T$  the nodes  $s$  and  $t$ , we get a parse tree  $T'$  with a yield  $w'$  that naturally satisfies  $\Delta^G \in \{\sum w', +\infty\}$ . Thus  $T'$  contradicts the minimality of  $T$ . We deduce that  $\Delta^G < +\infty$  and in particular  $\sum w = \Delta^G$ . Let us decompose the yield  $w$  as  $w = aubvc$ , where the subwords  $u, v$  derive from the pumpable path from node  $s$  to  $t$ . If  $\sum u + \sum v > 0$ , by inserting many copies of this subtree in  $T$ , we get  $\Delta^G = +\infty$ , which is impossible. It follows that  $\sum u + \sum v \leq 0$ . By collapsing the nodes  $s$  and  $t$ , we get a complete parse tree  $T'$  for  $G$  such that  $|T'| < |T|$  with a yield  $w'$  satisfying  $\sum w' = \sum w - (\sum u + \sum v)$ . From  $\sum u + \sum v \leq 0$  we derive  $\sum w' \geq \sum w$ . As  $\sum w = \Delta^G$  and  $\sum w' \leq \Delta^G$ , we derive  $\Delta^G = \sum w'$  and we get a contradiction on the minimality of  $T$ . It follows that  $T$  is elementary.  $\square$

**Lemma 5.3.** *If  $\Delta^G = +\infty$ , there exists a parse tree for  $G[X]$  where  $X$  is a non-terminal symbol derivable from the start symbol  $S$  with a yield  $uXv$  satisfying  $u, v \in A^*$ ,  $\sum uv > 0$ , and a number of nodes bounded by  $4^{|V|+1}$ .*

*Proof.* Let us first prove that there exists a non-terminal symbol  $X$  derivable from the start symbol  $S$  and a parse tree for  $G[X]$  with a yield  $uXv$  satisfying  $u, v \in A^*$  and  $\sum uv > 0$ . Since  $\Delta^G = +\infty$ , there exists a minimal (for the number of nodes) complete parse tree  $T$  with a yield  $w$  satisfying  $\sum w > 2^{|V|}$ . Observe that if  $T$  is elementary then  $|w| \leq 2^{|V|}$  and in particular  $\sum w \leq 2^{|V|}$  and we get a contradiction. So the tree  $T$  is not elementary. Hence there exists  $s \prec t$  in  $T$  with  $\text{sym}(s) = X = \text{sym}(t)$  for some non-terminal symbol  $X$ . The subtree of  $T$  between  $s$  and  $t$  provides a parse tree for  $G[X]$  with a yield  $uXv$  where  $u, v \in A^*$ . If  $\sum uv \leq 0$  by collapsing nodes  $s$  and  $t$  in  $T$ , we derive a complete parse tree  $T'$  such that  $|T'| < |T|$  with a yield  $w'$  satisfying  $\sum w' + \sum uv = \sum w$ . Thus  $\sum w' \geq 2^{|V|}$  and we get a contradiction on the minimality of  $|T|$ . Thus  $\sum uv > 0$ .

In the previous paragraph, we have proved that there exists a non-terminal symbol  $X$  derivable from the start symbol  $S$  and a parse tree  $T$  for  $G[X]$  with a yield  $uXv$  satisfying  $u, v \in A^*$  and  $\sum uv > 0$ . Without loss of generality, we can pick  $X$  and  $T$  is such a way  $|T|$  is minimal. Let  $t$  be the unique leaf of  $T$  labeled by  $X$  and assume by contradiction that  $|t| > |V|$ . In this case, there exists  $r \prec s \prec t$  such that  $\text{sym}(r) = X' = \text{sym}(s)$  for some non-terminal symbol  $X'$ . Notice that in that case the subtree between  $r$  and  $s$  is a parse tree  $T'$  for  $G[X']$  with a yield  $u'X'v'$  where  $u', v' \in A^*$ . Since  $|T'| < |T|$ , by minimality of  $T$ , we get  $\sum u'v' \leq 0$ . In particular, by collapsing in  $T$  the nodes  $s$  and  $t$  we get a parse tree  $T''$  for  $G[X]$  with a yield  $u''Xv''$  such that  $\sum u''v'' + \sum u'v' = \sum uv$ . Thus  $\sum u''v'' > 0$ . We get a contradiction on the minimality of  $|T|$  since  $|T''| < |T|$ . Hence  $|t| \leq |V|$ . Symmetrically, observe that if there exists  $r \prec s$  such that  $\text{sym}(r) = X' = \text{sym}(s)$  for some non-terminal symbol  $X'$  and such that  $r \not\prec t$  then we get a contradiction on the minimality of  $T$ . Therefore  $T$  can be decomposed as a branch for the

root to  $t$  in such a way nodes on this branch, except  $t$ , have exactly, on the right or on the left an elementary subtree. Thus  $|T| \leq |V| + 1 + |V|2^{|V|+1}$ . Since  $|V| + 1 \leq 2^{|V|+1}$ , we get  $|T| \leq 2^{|V|+1} + |V|2^{|V|+1} = (|V| + 1)2^{|V|+1} \leq 4^{|V|+1}$ .  $\square$

**Lemma 5.4.** *For every derivable non-terminal symbol  $X$ , there exists a parse tree with a yield in  $A^*XA^*$  and a number of nodes bounded by  $4^{|V|+1}$ .*

*Proof.* Since  $X$  is derivable from  $S$ , there exists a sequence  $X_0, \dots, X_k$  of non-terminal symbols with  $X_0 = S$ ,  $X_k = X$ ,  $k + 1 \leq |V|$ , and a sequence of production rules  $X_{j-1} \vdash \alpha_j X_j \beta_j$  with  $\alpha_j \beta_j = Y_j$  for some non-terminal symbol  $Y_j$ . Notice that there exists a complete elementary parse tree  $T_j$  for  $G[Y_j]$ . The parse trees  $T_1, \dots, T_k$  put along a branch labeled by  $X_0, \dots, X_k$  provide a parse tree for  $G$  with a yield in  $A^*XA^*$  and a number of nodes bounded by  $(k + 1) + k2^{|V|+1} \leq |V| + (|V| - 1)2^{|V|+1} \leq |V|2^{|V|+1} \leq 4^{|V|+1}$ .  $\square$

## A.6 Proofs for Section 6

*Fact 6.3.* It holds that  $|s| \leq |V|$  and  $|t| \leq |s| + |V| + 1$ .

*Proof.* Suppose, towards a contradiction, that  $|s| > |V|$ . There must exit two nodes  $p \prec q \prec s$  with  $\text{sym}(p) = \text{sym}(q)$ . If  $\text{in}(p) < \text{in}(q)$  then we get a certificate  $(\mathcal{T}', p, q)$  of strictly smaller rank than  $(\mathcal{T}, s, t)$  by setting the input value of  $t$  to  $-\infty$  and propagating onwards. If  $\text{in}(p) \geq \text{in}(q)$  then we may replace the subtree rooted in  $p$  by the subtree rooted in  $q$ , retaining the flow conditions since  $\text{out}(p) = \text{out}(q) = -\infty$  by [Fact 6.1](#), and thus get a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ . Both cases contradict the minimality of  $(\mathcal{T}, s, t)$ . This concludes the proof that  $|s| \leq |V|$ .

Now suppose, towards a contradiction, that  $|t| > |s| + |V| + 1$ . There exists necessarily two nodes  $s \prec p \prec q \prec t$  with  $\text{sym}(p) = \text{sym}(q)$ . If  $\text{in}(p) < \text{in}(q)$  then we get a certificate  $(\mathcal{T}', p, q)$  of strictly smaller rank than  $(\mathcal{T}, s, t)$  by setting the input value of  $t$  to  $-\infty$  and propagating onwards. Similarly, if  $\text{in}(p) = \text{in}(q)$  and  $\text{out}(q) < \text{out}(p)$  then we get a certificate  $(\mathcal{T}', p, q)$  of strictly smaller rank than  $(\mathcal{T}, s, t)$  by setting the output value of  $s$  to  $-\infty$  and propagating onwards. If  $\text{in}(p) = \text{in}(q)$  and  $\text{out}(q) \geq \text{out}(p)$  then we may replace the subtree rooted in  $p$  by the subtree rooted in  $q$ , and thus get a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ . The remaining case is when  $\text{in}(p) > \text{in}(q)$ . In that case, we collapse the nodes  $p \prec q$ , preserve the input value of  $p$ , and relabel all nodes lexicographically larger than  $p$  with the largest input/output values allowed by the flow conditions. In the resulting flow tree  $\mathcal{T}'$ , which has a smaller rank than  $\mathcal{T}$ , the node  $t'$  originating from  $t$  has a strictly larger input value than  $s$ . So  $(\mathcal{T}', s, t')$  is a certificate. All cases contradict the minimality of  $(\mathcal{T}, s, t)$ . This concludes the proof that  $|t| \leq |s| + |V| + 1$ .  $\square$

*Fact 6.4.* It holds that  $\text{in}(p) \leq \text{out}(p) + 2^{|V|}$  for every node  $p \in T$  with  $p \not\prec t$ .

*Proof.* Suppose, towards a contradiction, that  $in(p) > out(p) + 2^{|V|}$  for some node  $p \in T$  with  $p \not\preceq t$ . Recall that  $A = \{-1, 0, 1\}$  by assumption. If the subtree rooted in  $p$  has at most  $2^{|V|}$  leaves, then we may decrease the input and output values of its nodes, retaining the flow conditions, so that the output value of  $p$  is preserved and its new input value is at most  $out(p) + 2^{|V|}$ . Notice that this does not modify the main branch since  $p \not\preceq t$ . Thus, we get a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ . Otherwise, the subtree rooted in  $p$  has at least  $2^{|V|} + 1$  leaves. Hence, it is not elementary and we may reduce it into a strictly smaller, elementary subtree with at most  $2^{|V|}$  leaves. The latter induces a complete flow tree with the same input and output values for  $p$ . Again, this does not modify the main branch since  $p \not\preceq t$ . Thus, we get a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ .  $\square$

*Fact 6.5.* Let  $q \in T$  and let  $p$  be the parent of  $q$ . If  $p = t$  or  $p \not\preceq t$ , then  $out(q) \leq out(p) + 2^{|V|}$ . If moreover  $sym(p) = sym(q)$ , then  $out(q) < out(p)$ .

*Proof.* Assume that  $p = t$  or  $p \not\preceq t$ . Observe that the children of  $p$  are not on the main branch, i.e., none of them is a prefix of  $t$ . If  $q$  is the last child of  $p$ , then  $out(q) = out(p)$  by minimality of  $(\mathcal{T}, s, t)$ . Otherwise,  $q = p0$  and  $p1$  is the last child of  $p$ . It holds that  $out(p0) = in(p1)$  and  $out(p1) = out(p)$  by minimality of  $(\mathcal{T}, s, t)$ . We derive from [Fact 6.4](#) that  $out(q) \leq out(p) + 2^{|V|}$ .

Now assume, in addition, that  $sym(p) = sym(q)$ . Suppose, towards a contradiction, that  $out(q) \geq out(p)$ . If  $in(p) < in(q)$  then we get a certificate  $(\mathcal{T}', p, q)$  of strictly smaller rank than  $(\mathcal{T}, s, t)$  by setting the input value of  $t$  to  $-\infty$  and propagating onwards. If  $in(p) \geq in(q)$  then we may replace, retaining the flow conditions since  $out(q) \geq out(p)$ , the subtree rooted in  $p$  by the subtree rooted in  $q$ , and thus get a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ . Both cases contradict the minimality of  $(\mathcal{T}, s, t)$ . It follows that  $out(q) < out(p)$ , which concludes the proof of the fact.  $\square$

*Fact 6.6.* It holds that  $out(t) \leq out(s) \leq 6|V| \cdot 4^{|V|+1}$ .

*Proof.* If  $in(s) < in(t)$  then  $out(s) = out(t) = -\infty$  by [Fact 6.2](#), so the equality  $out(s) = out(t) + 1$  holds. Otherwise,  $out(t) < out(s)$ . If we had  $out(t) + 1 < out(s)$  then we could decrease the output value of  $s$  by one, retaining the flow conditions by [Fact 6.1](#), and get a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ , contradicting the minimality of  $(\mathcal{T}, s, t)$ . Therefore we get that

$$out(s) = out(t) + 1 \tag{3}$$

and in particular the first inequality of the claim.

Let us now prove that  $out(s) \leq K$ , where  $K \stackrel{\text{def}}{=} 3(|V| + 1)4^{|V|+1}$ . Suppose, towards a contradiction, that  $out(s) > K$ . Observe that  $out(t) \geq K$  because of [Equation \(3\)](#). Let us consider the subtrees on the right of the branch from  $s$  to  $t$ . The main idea of the proof is to replace these subtrees by smaller ones using [Theorem 5.2](#). Formally, let  $U = \{p1 \in T \mid s \preceq p \prec t \wedge p1 \not\preceq t\}$ . The set  $U$  collects the right-children of the main branch from  $s$  to the parent of  $t$ ,

excluding those that are on the branch themselves. Let  $u_1, \dots, u_k$  denote the elements of  $U$ , in lexicographic order, and let  $S_i = \text{sym}(u_i)$  for  $i = 1, \dots, k$ . Note that  $S_i \in V$  since  $G$  is in Chomsky normal form by assumption. Observe also that  $\text{out}(s) \leq \text{out}(t) + \Delta_{\#_1 \dots \#_k}$  due to the flow conditions. It follows from  $\text{out}(s) = \text{out}(t) + 1$  that  $\Delta_{\#_1 \dots \#_k} > 0$ .

If the total size of the subtrees rooted in  $u_1, \dots, u_k$  is at most  $K$ , then the flow conditions entail that the input and output values of their nodes are all strictly positive, since  $\text{out}(s) > K$  and  $A = \{-1, 0, 1\}$  by assumption. The same holds for the output values of the nodes  $p$  with  $s \preceq p \preceq t$ . So we may decrease all these values by one, retaining the flow conditions. Indeed, [Fact 6.1](#) guarantees that the first flow condition still holds for the parent of  $s$ . We obtain, in this way, a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ .

Otherwise, the total size of the subtrees rooted in  $u_1, \dots, u_k$  is at least  $K + 1$ . Observe that  $k \leq |V| + 1$  by [Fact 6.3](#). According to [Theorem 5.2](#), there exists  $T_1, \dots, T_k$ , where each  $T_i$  is a complete parse tree for  $G[S_i]$  with yield  $z_i$ , such that  $|T_1| + \dots + |T_k| \leq 3k4^{|V|+1} \leq K$  and  $\sum z_1 \dots z_k > 0$ . Let us replace the subtrees rooted in  $u_1, \dots, u_k$  by  $T_1, \dots, T_k$ , respectively. Since  $\text{out}(t) \geq K$ , this induces a complete flow tree  $\mathcal{T}' = (T', \text{sym}', \text{in}', \text{out}')$ , with  $\text{out}'(t) = \text{out}(t)$ , and satisfying  $\text{out}'(s) = \text{out}'(t) + \sum z_1 \dots z_k > \text{out}'(t)$ . The new output value of  $s$  might be smaller, but [Fact 6.1](#) guarantees that the first flow condition still holds for the parent of  $s$ . The input values of  $s$  and  $t$  were not changed, so we have  $\text{in}'(s) = \text{in}(s) \leq \text{in}(t) = \text{in}'(t)$ . Therefore,  $(\mathcal{T}', s, t)$  is a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ .

In both cases, we obtain a contradiction with the minimality of  $(\mathcal{T}, s, t)$ . The observation that  $K \leq 6|V| \cdot 4^{|V|+1}$  concludes the proof of the fact.  $\square$

*Fact 6.7.* It holds that  $\text{in}(s) \leq \text{in}(t) \leq 7|V| \cdot 4^{|V|+1}$ .

*Proof.* We first show that  $\text{in}(s) \leq \text{in}(t) \leq \text{in}(s) + 1$ . Recall that  $\text{in}(s) \leq \text{in}(t)$  by definition of certificates. If we had  $\text{in}(s) + 1 < \text{in}(t)$  then we could decrease the input value of  $t$  by one, retaining the flow conditions by [Fact 6.2](#), and get a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ , contradicting the minimality of  $(\mathcal{T}, s, t)$ . Therefore,  $\text{in}(t) \leq \text{in}(s) + 1$ .

Observe that  $t$  is an internal node since  $\text{sym}(s) = \text{sym}(t)$  cannot be in  $A \cup \{\varepsilon\}$ . Let us bound the input value of its first child. According to [Fact 6.4](#), it holds that  $\text{in}(tj) \leq \text{out}(tj) + 2^{|V|}$  for each child  $tj$  of  $t$ . Let  $k \in \{1, 2\}$  denote the number of children of  $t$ . The flow conditions together with the minimality of  $(\mathcal{T}, s, t)$  guarantee that  $\text{in}(t0) = \text{in}(t)$ ,  $\text{out}(t) = \text{out}(t(k-1))$ , and  $\text{in}(t(j+1)) = \text{out}(tj)$  for every  $j = 0, \dots, k-1$ . We derive that  $\text{in}(t0) \leq \text{out}(t) + 2 \cdot 2^{|V|}$ . It follows from [Fact 6.6](#) that  $\text{in}(t0) \leq K + 2^{|V|+1}$ , where  $K \stackrel{\text{def}}{=} 6|V| \cdot 4^{|V|+1}$ .

Let us now prove that  $\text{in}(t) \leq H$ , where  $H \stackrel{\text{def}}{=} K + 2^{|V|+1}$ . The proof is similar to the proof of [Fact 6.6](#). Suppose, towards a contradiction, that  $\text{in}(t) > H$ . Observe that  $\text{in}(s) \geq H$ . Let us consider the subtrees on the left of the branch from  $s$  to  $t$ . The main idea of the proof is to replace these subtrees by smaller ones using [Theorem 5.2](#). Formally, let  $U = \{p0 \in T \mid s \preceq p \prec t \wedge p0 \not\preceq t\}$ . The set  $U$  collects the left-children of the main branch from  $s$  to the parent of  $t$ ,

excluding those that are on the branch themselves. Let  $u_1, \dots, u_k$  denote the elements of  $U$ , in lexicographic order, and let  $S_i = \text{sym}(u_i)$  for  $i = 1, \dots, k$ . Note that  $S_i \in V$  since  $G$  is in Chomsky normal form by assumption. Observe also that  $\text{in}(t) \leq \text{in}(s) + \Delta_{\#_1 \dots \#_k}$ . It follows that  $\Delta_{\#_1 \dots \#_k} \geq 0$  and that  $\Delta_{\#_1 \dots \#_k} > 0$  if  $\text{in}(s) < \text{in}(t)$ .

If the total size of the subtrees rooted in  $u_1, \dots, u_k$  is at most  $K$ , then the flow conditions entail that the input and input values of their nodes are all strictly positive, since  $\text{in}(t) > H \geq K$  and  $A = \{-1, 0, 1\}$  by assumption. The same holds for the input values of the nodes  $p$  with  $s \preceq p \preceq t$ . So we may decrease all these values by one, retaining the flow conditions. Indeed, the first flow condition still holds for  $t$  since  $\text{in}(t0) \leq H$ . We obtain, in this way, a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ .

Otherwise, the total size of the subtrees rooted in  $u_1, \dots, u_k$  is at least  $K + 1$ . Observe that  $k \leq |V| + 1$  by [Fact 6.3](#). According to [Theorem 5.2](#), there exists  $T_1, \dots, T_k$ , where each  $T_i$  is a complete parse tree for  $G[S_i]$  with yield  $z_i$ , such that  $|T_1| + \dots + |T_k| \leq 3k4^{|V|+1} \leq K$  and  $\sum z_1 \dots z_k \geq 0$ . Moreover  $\sum z_1 \dots z_k = 0$  only if  $\text{in}(s) = \text{in}(t)$ . Let us replace the subtrees rooted in  $u_1, \dots, u_k$  by  $T_1, \dots, T_k$ , respectively. Since  $\text{in}(s) \geq H \geq K$ , this induces a complete flow tree  $\mathcal{T}' = (T', \text{sym}', \text{in}', \text{out}')$ , with  $\text{in}'(s) = \text{in}(s)$ , and satisfying  $\text{in}'(t) = \text{in}'(s) + \sum z_1 \dots z_k \geq \text{in}'(s)$ . The first flow condition still holds for  $t$  since  $\text{in}'(t0) = \text{in}(t0) \leq H \leq \text{in}(s) = \text{in}'(s) \leq \text{in}'(t)$ . The output values of  $s$  and  $t$  were not changed. It follows that  $\text{in}'(s) < \text{in}'(t)$  or  $\text{out}'(t) < \text{out}'(s)$ . Indeed, if  $\text{in}'(s) = \text{in}'(t)$  then  $\sum z_1 \dots z_k = 0$ , hence,  $\text{in}(s) = \text{in}(t)$ , which entails that  $\text{out}'(t) = \text{out}(t) < \text{out}(s) = \text{out}'(s)$ . Therefore,  $(\mathcal{T}', s, t)$  is a certificate of strictly smaller rank than  $(\mathcal{T}, s, t)$ .

In both cases, we obtain a contradiction with the minimality of  $(\mathcal{T}, s, t)$ . The observation that  $H \leq 7|V| \cdot 4^{|V|+1}$  concludes the proof of the fact.  $\square$

*Fact 6.8.* It holds that  $\text{out}(p) \leq 4^{2(|V|+1)}$  for every ancestor  $p \preceq t$ .

*Proof.* Let us write  $t = sj_1 \dots j_k$  where each  $j_i \in \{0, 1\}$ , and let  $p_i = sj_1 \dots j_i$  for  $i = 0, \dots, k$ . We first show that for every  $0 < i \leq k$ ,

$$\text{out}(p_i) \leq \text{out}(p_{i-1}) + 2^{|V|}. \quad (4)$$

Indeed, the flow conditions together with the minimality of  $(\mathcal{T}, s, t)$  guarantee, for every ancestor  $s \preceq p \prec t$ , that  $\text{out}(p) = \text{out}(p1)$  and that  $\text{out}(p0) = \text{in}(p1)$  if  $p0 \preceq t$ . In the latter case,  $\text{in}(p1) \leq \text{out}(p1) + 2^{|V|}$  by [Fact 6.4](#), hence,  $\text{out}(p0) \leq \text{out}(p) + 2^{|V|}$ . We have thus shown that  $\text{out}(pj) \leq \text{out}(p) + 2^{|V|}$  for every ancestor  $s \preceq p \prec t$  and every  $j \in \{0, 1\}$  such that  $pj \preceq t$ .

We derive from [Equation \(4\)](#) that  $\text{out}(p_i) \leq \text{out}(s) + i2^{|V|}$  for all  $0 < i < k$ . Recall that  $k \leq |V| + 1$  by [Fact 6.3](#). It follows from [Fact 6.6](#) that for every node  $p$  such that  $s \preceq p \preceq t$ , we have

$$\text{out}(p) \leq \text{out}(s) + |V| \cdot 2^{|V|} \leq 6|V| \cdot 4^{|V|+1} + |V| \cdot 2^{|V|} \leq 4^{2(|V|+1)}$$

According to [Fact 6.1](#), it holds that  $\text{out}(p) = -\infty$  for every proper ancestor  $p \prec s$ , which concludes the proof of the fact.  $\square$

*Fact 6.9.* It holds that  $\text{in}(p) \leq 4^{2(|V|+1)}$  for every ancestor  $\varepsilon \prec p \preceq t$ .

*Proof.* Let us write  $t = j_1 \cdots j_k$  where each  $j_i \in \{0, 1\}$ , and let  $p_i = j_1 \cdots j_i$  for  $i = 1, \dots, k$ . We claim that  $\text{in}(p_{i-1}) \leq \text{in}(p_i) + 2^{|V|}$  for every  $1 < i \leq k$ . Indeed, the flow conditions together with the minimality of  $(\mathcal{T}, s, t)$  guarantee, for every ancestor  $\varepsilon \prec p \prec t$ , that  $\text{in}(p) = \text{in}(p0)$  and that  $\text{out}(p0) = \text{in}(p1)$  if  $p1 \preceq t$ . In the latter case,  $\text{in}(p0) \leq \text{out}(p0) + 2^{|V|}$  by [Fact 6.4](#), hence,  $\text{in}(p) \leq \text{in}(p1) + 2^{|V|}$ . We have thus shown that  $\text{in}(p) \leq \text{in}(pj) + 2^{|V|}$  for every ancestor  $\varepsilon \prec p \prec t$  and every  $j \in \{0, 1\}$  such that  $pj \preceq t$ . This concludes the proof of the claim.

Observe that  $s = j_1 \cdots j_h$  where  $h = |s|$ . We derive from the claim that, firstly,  $\text{in}(p_i) \leq \text{in}(s) + (h - i)2^{|V|}$  for all  $0 < i < h$ , and secondly,  $\text{in}(p_i) \leq \text{in}(t) + (k - i)2^{|V|}$  for all  $h < i < k$ . Recall that  $h \leq |V|$  and  $(k - h) \leq |V| + 1$  by [Fact 6.3](#). It follows from [Fact 6.7](#) that, for every ancestor  $\varepsilon \prec p \preceq t$ , we have

$$\text{in}(p) \leq \max\{\text{in}(s), \text{in}(t)\} + |V|2^{|V|} \leq 7|V| \cdot 4^{|V|+1} + |V|2^{|V|} \leq 4^{2(|V|+1)}$$

which concludes the proof of the fact.  $\square$